MAT 555 Industrial Mathematics: Continuous Models Practicum

Pennes Bioheat Equation

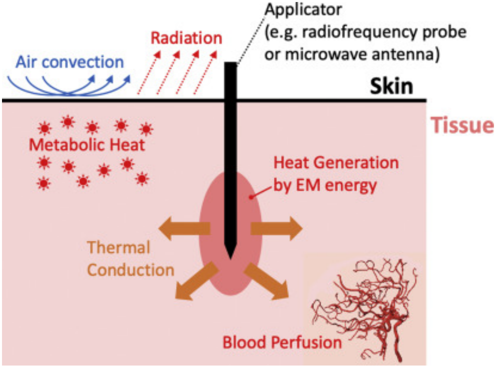
Johnathan Makar

Department of Mathematics
West Chester University of Pennsylvania

4/17/2024

Pennes Experiment

Pennes goal was to evaluate the applicability of heat flow theory to the forearm in basic terms of local rate of tissue heat production and volume flow of blood



Physiological Derivation

Fick's principle:

$$h_b = V \cdot s(\theta_a - \theta_v) \tag{1.1}$$

Newtons Law of Cooling applied to the forearm:

$$\theta = \left[\theta_a - \frac{(h_m + h_b)}{4K} R^2\right] - \frac{(h_m + h_b)}{4K} r^2 \tag{1.2}$$

V	Volume flow of blood	s	Specific heat of blood
h_b	Rate of heat transfer, blood to tissue	θ_v	Venous blood temperature
θ_a	Arterial blood temperature	K	Specific thermal conductivity of tissue
h_m	rate of tissue heat production	r	Radial distance from the axis
R	Radius of the cylinder		

Pennes' Assumptions

- The cross-section of a forearm is cylindrical
- The rate of heat production by tissue will be considered uniform throughout the forearm
- The volume flow of blood is constant
- The specific thermal conductivity K will be uniform

Using his assumptions and prior equations, Pennes established the general heat equation in cylindrical coordinates:

$$cp\frac{\partial\theta}{\partial t} = -K\left[\frac{\partial^2\theta}{\partial r^2} + \frac{1}{r}\frac{\partial\theta}{\partial r} + \frac{1}{r^2}\frac{\partial\theta}{\partial\phi} + \frac{\partial^2\theta}{\partial Z^2}\right] + h_m + h_b \quad (1.3)$$

c	Coefficient of heat for tissue	p	Density of tissue
K	Specific thermal conductivity of tissue	r	Radial distance from the axis
ϕ	Angular gradient	Z	Long axis gradient
θ	Tissue temperature	h_m	Rate of tissue heat production
h_b	Rate of heat transfer, blood to tissue		

Pennes' Phenomenon

- Pennes paper received criticism due to his experimental data and his modeled data not matching up
- Wissler found that Pennes had been using certain parameters that were available at the time, but not accurate for the human body
- Wissler found that more standardized parameters could be applied to get the model output closer to experimental data
- The result is that temperature profiles computed by using the Pennes model agree with the measured profiles as well as can be expected

Pennes Bioheat Equation

Pennes did not have the computer power back in 1948 to deal with cartesian coordinates, so we converted the cylindrical into cartesian:

$$cp\frac{\partial\theta}{\partial t} = -K\left[\frac{\partial^2\theta}{\partial x^2} + \frac{1}{r}\frac{\partial^2\theta}{\partial y^2} + \frac{\partial^2\theta}{\partial z^2}\right] + h_m + h_b \qquad (1.4)$$

c	Coefficient of heat for tissue	p	Density of tissue
K	Specific thermal conductivity of tissue	θ	Tissue temperature
h_m	Rate of tissue heat production	h_b	Rate of heat transfer, blood to tissue

1D Pennes Bioheat Equation

From Pennes assumptions of equilibrium, we can derive the 1-D equation to be:

$$cp\frac{\partial \theta}{\partial t} = K\frac{\partial^2 \theta}{\partial x^2} + V \cdot s(K-1)(\theta - \theta_v) + h_m \tag{1.5}$$

subject to

$$\begin{aligned} \text{Initial condition:} \quad & \theta(x,0) = \theta_0(x) \\ \text{Boundary condition:} \quad & \theta(0,t) = \theta(L,t) \end{aligned}$$

Analytical Result

$$\theta(r^*) = \theta_{\text{inf}} + (\theta_a - \theta_{\text{inf}})(1 + \frac{q_m^*}{w_b^*}) \left(\frac{I_0(\sqrt{w_b^*} r^*)}{1 - I_0(\sqrt{w_b^*}) + \frac{\sqrt{w_b^*}}{h_A^*} I_1(\sqrt{w_b^*})} \right)$$

Where I_0 and I_1 are special functions called Bessel Functions.

q_m	Metabolic heat generation per unit volume
w_b	Blood perfusion rate
h_A	Coefficient of convection and radiation

Numerical Solution

- Since we were able to get an analytical solution in the 1-D case, we worked on getting a numerical solution in the same case
- We used the Forward Difference Method in the time direction and the Central Difference Method in the x-direction. This allows us to get the updating equation below:

$$\theta_i^{n+1} = \left(1 - \frac{2\alpha\Delta t}{\Delta x^2}\right)\theta_i^n + \alpha\frac{\Delta t}{\Delta x^2}\left(\theta_{i+1}^n + \theta_{i-1}^n\right) + g_i^n\Delta t \tag{1.6}$$

where $g_i^n = \frac{V \cdot s(K-1)(\theta_i^n - \theta_a + h_m)}{cp}$ and $\alpha = \frac{K}{cp}$
for $n = 0, 1, 2, ..., n_t$ and $i = 1, 2, ..., n_x$

Experimental Results

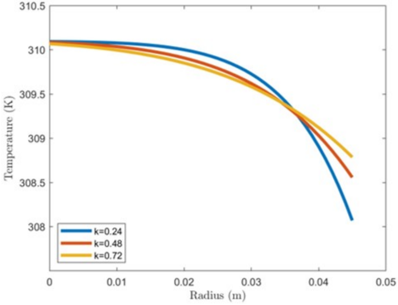


Figure: K Parameter Testing

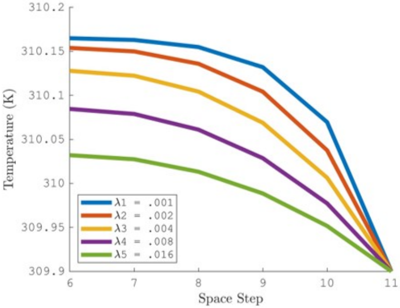


Figure: λ Parameter Testing

where $\lambda = \frac{K}{cp}\frac{\Delta t}{\Delta x^2}$

Experimental Results

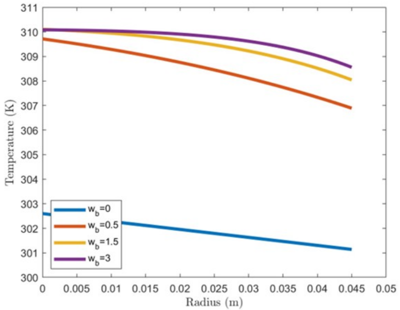


Figure: Blood Perfusion Rate Testing

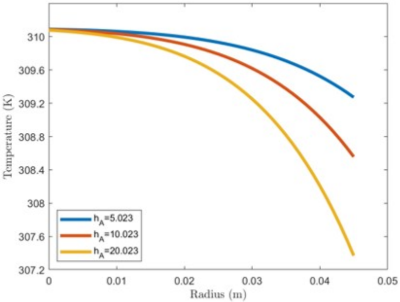


Figure: Coefficient of Heat Transfer

w_b	Blood perfusion rate
h_A	Coefficient of convection and radiation

Conclusion

- The 1-D Bioheat Equation and its many parameters can be tested to see which parameter most significantly changes the result
- This research is a stepping stone for a new cancer treatment method called Magnetic Fluid Hyperthermia
- Human bodies are unique and each treatment would be case-by-case. Studying how the parameters affect the heat propagation will help doctors create the best treatment method

Further Research

- Work towards solving the 2-D and 3-D Bioheat Equation
- Continue parameter testing in different locations inside the body
- Apply the Bioheat Equation to irregular domains

References

1. Pennes, Harry H. “Analysis of Tissue and Arterial Blood Temperatures in the Resting Human Forearm.” Journal of Applied Physiology, vol. 1, no. 2, Aug. 1948, pp. 93–122
2. Wissler, Eugene H. “Pennes’ 1948 paper revisited.” Journal of Applied Physiology, vol. 85, no. 1, 1 July 1998, pp. 35–41.
3. Yue, Kai, et al. “An Analytic Solution of One-dimensional Steady-state Pennes’ Bioheat Transfer Equation in Cylindrical Coordinates.” Journal of Thermal Science, vol. 13, no. 3, Aug. 2004, pp. 255–258.