The topology of path component spaces

Jeremy Brazas

October 26, 2012

Abstract

The path component space of a topological space X is the quotient space of X whose points are the path components of X. This paper contains a general study of the topological properties of path component spaces including their relationship to the zeroth dimensional shape group.

Path component spaces are simple-to-describe and well-known objects but only recently have recieved more attention. This is primarily due to increased interest and application of the quasitopological fundamental group $\pi_1^{qtop}(X, x_0)$ of a space X with basepoint x_0 and its variants; See e.g. [2, 3, 5, 7, 8, 9]. Recall $\pi_1^{qtop}(X, x_0)$ is the path component space of the space $\Omega(X, x_0)$ of loops based at $x_0 \in X$ with the compact-open topology. The author claims little originality here but knows of no general treatment of path component spaces.

1 Path component spaces

Definition 1. The *path component space* of a topological space X, is the quotient space $\pi_0(X)$ obtained by identifying each path component of X to a point.

If $x \in X$, let $[x]$ denote the path component of x in X. Let $q_X : X \to \pi_0(X)$, $q(x) = [x]$ denote the canonical quotient map. We also write $[A]$ for the image $q_X(A)$ of a set $A \subset X$. Note that if $f : X \to Y$ is a map, then $f([x]) \subseteq [f(x)]$. Thus f determines a well-defined function $f_0 : \pi_0(X) \to \pi_0(Y)$ given by $f_0([x]) = [f(x)]$. Moreover, f_0 is continuous since $f_0 q_X = q_Y f$ and q_X is quotient. Altogether, we obtain an endofunctor of the category **Top** of topological spaces.

Proposition 2. $\pi_0 : \mathbf{Top} \to \mathbf{Top}$ *is a functor.*

Remark 3. If X has a given basepoint x_0, we take the basepoint of $\pi_0(X)$ to be $[x_0]$ and denote the resulting based space as $\pi_0(X, x_0)$. This gives a functor on the category of based topological spaces.

Definition 4. A space X is *semi-locally* 0-*-connected* if for every point $x \in X$, there is a neighborhood U of x such that the inclusion $i : U \to X$ induces the constant map $i_0 : \pi_0(U) \to \pi_0(X)$.

Note that $\pi_0(X)$ is discrete if and only if each path component of X is open.

Proposition 5. *$\pi_0(X)$ is discrete iff X is semi-locally 0-connected.*

Proof. If X is semi-locally 0-connected and $x \in X$, then there is a neighborhood U of x such that the inclusion $i : U \to X$ induces the constant map $i_0 : \pi_0(U) \to \pi_0(X)$. Thus $U \subseteq [x]$ and $[x]$ is open in X. Since any given path component of X is open, $\pi_0(X)$ is discrete. Conversely, suppose $\pi_0(X)$ is discrete. If $x \in X$, let U be the open set $[x]$. Clearly the inclusion $[x] \to X$ induces the constant map $\pi_0([x]) \to \pi_0(X)$. □

Corollary 6. *If X is locally path connected, then $\pi_0(X)$ is discrete.*

Example 7. If X is a geometric simplicial complex or a CW-complex, then $\pi_0(X)$ is discrete.

Another case of interest is when the natural quotient map $q_X : X \to \pi_0(X)$ is a homeomorphism.

Remark 8. The quotient map $q_X : X \to \pi_0(X)$ is a homeomorphism iff it is injective. Thus q_X is a homeomorphism iff X is totally path disconnected (i.e. each path component consists of a single point). Specifically, $X \cong \pi_0(X)$ if X is totally disconnected or zero dimensional (there is a basis for the topology of X consisting of clopen sets).

Another general and useful fact about path component spaces is that every space is a path component space.

Theorem 9. *[10] For every space X, there is a paracompact Hausdorff space $H(X)$ and a natural homeomorphism $\pi_0(H(X)) \cong X$.*

2 Examples

Example 10. Let $\mathbb{T} \subseteq \mathbb{R}^2$ be the topologist's sine curve

$$\mathbb{T} = \{(0,0)\} \cup \left\{(x,y) | y = \sin\left(\frac{1}{x}\right), 0 < x \leq 1/\pi\right\}$$

and $\mathbb{T}_c = \mathbb{T} \cup \{0\} \times [-1,1]$ be the closed topologist's sine curve. It is easy to see that in both cases, the path component space $\pi_0(\mathbb{T}) \cong \pi_0(\mathbb{T}_c)$ is homeomorphic to the *Sierpinski space* $\mathbb{S} = \{0,1\}$ with topology $\{\emptyset, \{1\}, \{0,1\}\}$. In particular, the open set $\{1\}$ cooresponds to the path component of $(1/\pi, 0)$.

The above example allows us to characterize when certain path component spaces are T_1. Recall that a space X is *sequential* if for every non-closed set $A \subset X$, there is a convergent sequence $x_n \to x$ such that $x_n \in A$ and $x \notin A$. It is well-known that metric spaces are sequential and sequential spaces are precisely those which are quotients of metric spaces.

Proposition 11. *If $\pi_0(X)$ is T_1, then any map $f : \mathbb{T} \to X$ from the topologist's sine curve induces the constant map $f_0 : \mathbb{S} \to \pi_0(X)$. If X is sequential, the converse holds.*

Proof. Let $A = \{(0,0)\}$ and $B = \mathbb{T}\backslash A$ be the path components so that $\pi_0(\mathbb{T}) = \{A, B\}$ with topology $\{\emptyset, \{B\}, \{A, B\}\}$ and let $t_n = \frac{1}{n\pi}$, $n \geq 1$. Suppose $\pi_0(X)$ is T_1 and $f : \mathbb{T} \to X$ be a map. Since the singleton $\{f(B)\}$ is closed in X, $f_0^{-1}(f(B))$ is non-empty and closed in $\pi_0(\mathbb{T})$. Since the only non-empty closed set in $\pi_0(\mathbb{T})$ containing B is $\pi_0(\mathbb{T}) = \{A, B\}$, we have $f_0^{-1}(f(B)) = \{A, B\}$. Therefore $f_0(A) = f_0(B)$ showing f_0 is constant.

Suppose X is sequential and $x \in X$. To obtain a contradiction, suppose the singleton $\{[x]\}$ is not closed in $\pi_0(X)$. Since $q_X : X \to \pi_0(X)$ is quotient, $[x]$ is not closed in X. Thus there is a convergent sequence $x_n \to x$ such that $x_n \in [x]$ and $x \notin [x]$. Define a function $f : \mathbb{T} \to X$ by $f(t_n) = x_n$ and $f(0,0) = x$. Extend f to the rest of $\mathbb{T}$ by defining f on the arc from t_n to t_{n+1} to be a path from x_n to x_{n+1} in X (which exists since $x_n \in [x]$ for all n). Clearly f is continuous and induces an injection $f_0 : \mathbb{S} \to \pi_0(X)$. □

One can generalize this approach to general spaces by replacing convergent sequences with convergent nets and $\mathbb{T}$ by analogous constructions based on directed sets.

The next example allows us to realiize subspaces of $\mathbb{R}$ as path component spaces of simple spaces constructed independently of Theorem 9.

Example 12. Let X be the set $\mathbb{R} \times [0,1]$. We define a Hausdorff topology on X such that $\pi_0(A \times [0,1]) \cong A$ for each subset of the form $A \times [0,1] \subseteq X$. The topology on X has a basis consisting of sets of the form $\{a\} \times (s,t)$ and $\{a\} \times (t,1] \cup (a,b) \times [0,1] \cup \{b\} \times [0,s)$ for $0 < s < t < 1$ and $a < b$. This topology is a simple extension of the *ordered square* in [14, §16, Example 3] and is the order topology given by the dictionary ordering on X. The path components of X are $\{z\} \times [0,1]$ for $z \in \mathbb{R}$ (see [14, §24, Example 6]). It then suffices to show that for each $A \subseteq \mathbb{R}$, the projection $p_A : X_A \to A$ is quotient, where $X_A = A \times [0,1]$ has the subspace topology of X. Suppose U is open in $\mathbb{R}$ so that $U \cap A$ is open in A. Since $U \times [0,1] = p_{\mathbb{R}}^{-1}(U)$ is open in X and so $(U \times [0,1]) \cap X_A = (U \cap A) \times [0,1] = p_A^{-1}(U \cap A)$ is open in X_A. Therefore p_A is continuous. Now suppose $V \subseteq A$ such that $p_A^{-1}(V) = V \times [0,1]$ is open in X_A. For each $v \in V$, there is an open neighborhood $\{v\} \times (t_v, 1] \cup (v, b_v) \times I \cup \{b_v\} \times [0, s_v)$ of $(v,1)$ contained in $V \times [0,1]$. Since $V \times [0,1]$ is saturated with respect to p_A, we have $([v, b_v] \cap A) \times [0,1] \subset V \times [0,1]$. Similarly, since $(v,0) \in V$ for each $v \in V$ we can find a closed interval $[a_v, v]$ such that $([a_v, v] \cap A) \times [0,1] \subset V \times [0,1]$. Therefore, for each $v \in V$, we have $v \in (a_v, b_v) \cap A \subseteq V$. Therefore V is open in A. Thus p_A is quotient, and consequently $\pi_0(X_A) \cong A$.

Example 13. Using the previous example, we can find a smiple space Y such that $\pi_0(Y) \cong S^1$. Let $\epsilon : \mathbb{R} \to S^1$ denote the exponential map and $X = \mathbb{R} \times [0,1]$ be the space defined in the previous example. Let Y be the set $S^1 \times [0,1]$ with the quotient topology with respect to $\epsilon \times id : X \to S^1 \times [0,1]$. The projection $Y \to S^1$ is precisely the quotient map $q_Y : Y \to \pi_0(Y)$ and so $S^1 \cong \pi_0(Y)$.

3 Limits and colimits

Proposition 14. $\pi_0 : \mathbf{Top} \to \mathbf{Top}$ *preserves coproducts and quotients.*

Proof. It is immediate that $\pi_0(\coprod_\lambda X_\lambda) \cong \coprod_\lambda \pi_0(X_\lambda)$. If $p : X \to Y$ is a quotient map, then $q_Y \circ p = p_0 \circ q_X$ is a quotient map. Since q_X is quotient, $p_0 = \pi_0(p)$ is quotient. □

Though π_0 preserves coproducts, unfortunately it fails to be cocontinuous (in the sense that it preserves all colimits). Since **Top** is cocomplete, it suffices to exhibit a coequalizer which is not preserved [11].

Example 15. Let $Y = \left\{1, \frac{1}{2}, \frac{1}{3}, ..., 0\right\} \subseteq \mathbb{R}$. We define parallel maps $f, g : \mathbb{N} \to Y$ by $f(n) = \frac{1}{n}$ and $g(n) = \frac{1}{n+1}$. It is easy to see that the coequalizer of these maps is homeomorphic to the Sierpinski space $\mathbb{S} = \{0, 1\}$ of Example 10. The Sierpinski space is path connected since the function $\alpha : [0, 1] \to \{0, 1\}$ given by $\alpha([0, \frac{1}{2}]) = 0$ and $\alpha((\frac{1}{2}, 1]) = 1$ is continuous. Therefore $\pi_0(\mathbb{S})$ is a one point space. Noting that both $\mathbb{N} = \{1, 2, ..., \}$ and Y are totally path disconnected (so $\pi_0(\mathbb{N}) \cong \mathbb{N}$ and $\pi_0(Y) \cong Y$), we find that $f = f_0$ and $g = g_0$. Thus the coequalizer of f_0 and g_0 is $\mathbb{S}$ which is not a one point space. It follows that the path component space of the coequalizer of f and g is not homeomorphic to the coequalizer of f_0 and g_0.

One might notice in the previous example that the path component space of the coequalizer is a quotient of the coequalizer of the induced maps. This phenomenon in fact generalizes to all (small) colimits.

Proposition 16. *Let J be a small category and $F : J \to$ **Top** be a diagram with colimit colimF. Suppose colim$(\pi_0 \circ F)$ is the colimit of $\pi_0 \circ F : J \to$ **Top**. There is a canonical quotient map Q : colim$(\pi_0 \circ F) \to \pi_0($colimF$)$.*

Proof. By the colimit existence theorem [11, §V.4], *colimF* is the coequalizer of parallel maps f and g and *colim*$(\pi_0 \circ F)$ is the coequalizer of parallel maps f' and g' as seen in the diagram below. The coproducts on the left are over all morphisms $u : j \to k$ in J and the coproducts in the middle column are over all objects $i \in J$. The naturality of $q_X : X \to \pi_0(X)$ and the homeomorphisms $\pi_0(\coprod_\alpha X_\lambda) \cong \coprod_\lambda \pi_0(X_\lambda)$ of Proposition 14 gives the commutativity of the squares on the left and top right. By Proposition 14, r_0 is a quotient map. Since $r \circ f = r \circ g$, we have $r_0 \circ f_0 = r_0 \circ g_0$. Therefore $r_0 \circ t \circ f' = r_0 \circ f_0 \circ s = r_0 \circ g_0 \circ s = r_0 \circ t \circ g'$. By the universal property of *colim*$(\pi_0 \circ F)$, this induces a unique map Q : *colim*$(\pi_0 \circ F) \to \pi_0($*colimF*$)$ such that $Q \circ r' = r_0 \circ t$. Since t is a homeomorphism and r_0 is a quotient map, Q is also quotient.

$$\begin{tikzcd}
\coprod_{u:j\to k} F(j) \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] \arrow[d, "q"'] & \coprod_{i\in J} F(i) \arrow[r, "r"] \arrow[d, "q"'] & colimF \arrow[d, "q"'] \\
\pi_0\left(\coprod_{u:j\to k} F(j)\right) \arrow[r, shift left, "f_0"] \arrow[r, shift right, "g_0"'] & \pi_0\left(\coprod_{i\in J} F(i)\right) \arrow[r, "r_0"] & \pi_0(colimF) \\
\coprod_{u:j\to k} \pi_0(F(j)) \arrow[r, shift left, "f'"] \arrow[r, shift right, "g'"'] \arrow[u, "s"', "\cong"] & \coprod_{i\in J} \pi_0(F(i)) \arrow[r, "r'"] \arrow[u, "t"', "\cong"] & colim(\pi_0 \circ F) \arrow[u, dashed, "\exists! Q"']
\end{tikzcd}$$

□

Corollary 17. *Let $X \cup_Z Y$ be the pushout of the diagram $X \xleftarrow{f} Z \xrightarrow{g} Y$ where $\pi_0(Y)$ is discrete and $g_0 : \pi_0(Z) \to \pi_0(Y)$ is surjective. The inclusion $j : X \to X \cup_Z Y$ induces a quotient map $j_0 : \pi_0(X) \to \pi_0(X \cup_Z Y)$ on path component spaces.*

We now observe the behavior of π_0 on products.

Proposition 18. *Let $\{X_\lambda\}$ be a family of spaces and $X = \prod_\lambda X_\lambda$. Let $q_\lambda : X_\lambda \to \pi_0(X_\lambda)$ and $q_X : X \to \pi_0(X)$ be the canonical quotient maps and $\prod_\lambda \pi_\lambda : X \to \prod_\lambda \pi_0(X_\lambda)$ be the product map. There is a natural continuous bijection $\Phi : \pi_0(X) \to \prod_\lambda \pi_0(X_\lambda)$ such that $\Phi \circ q_X = \prod_\lambda q_\lambda$.*

Proof. The projections $pr_\lambda : X \to X_\lambda$ induces maps $(pr_\lambda)_0 : \pi_0(X) \to \pi_0(X_\lambda)$ which in turn induce the natural map $\Phi : \pi_0(X) \to \prod_\lambda \pi_0(X_\lambda)$, $\Phi([(x_\lambda)]) = ([x_\lambda])$. Clearly Φ is surjective. If $[x_\lambda] = [y_\lambda]$ for each λ, then there is a path $\alpha_\lambda : [0,1] \to X_\lambda$ from x_λ to y_λ. These maps induce a path $\alpha : [0,1] \to X$ such that $pr_\lambda \circ \alpha = \alpha_\lambda$. Since α is a path from (x_λ) to (y_λ), we have $[(x_\lambda)] = [(y_\lambda)]$. Thus Φ is injective. □

Corollary 19. *$\Phi : \pi_0(X) \to \prod_\lambda \pi_0(X_\lambda)$ is a homeomorphism if and only if the product of quotients $\prod_\lambda q_\lambda : X \to \prod_\lambda \pi_0(X_\lambda)$ is itself a quotient map.*

Proof. This follows from the fact that q_X is a quotient map, Φ is a bijection, and $\Phi \circ q_X = \prod_\lambda q_\lambda$. □

Corollary 20. *If $\pi_0(X_\lambda)$ is discrete for each λ, then $\Phi : \pi_0(X) \to \prod_\lambda \pi_0(X_\lambda)$ is a homeomorphism.*

Proof. If $\pi_0(X_\lambda)$ is discrete, then $\pi_\lambda : X_\lambda \to \pi_0(X_\lambda)$ is open. Since products of open maps are open, $\prod_\lambda \pi_\lambda : X \to \prod_\lambda \pi_0(X_\lambda)$ is open and must be quotient. By Corollary 19, Φ is a homeomorphism. □

Corollary 21. *π_0 does not preserve finite products.*

Proof. Let $\mathbb{R}_K$ be the real line with topology generated by the sets $(a,b), (a,b)\backslash K$ where $K = \{1, 1/2, 1/3, 1/4, ...\}$. Let $\mathbb{Q}_K$ be the rational numbers with the subspace topology of $\mathbb{R}_K$. Now let $X = \mathbb{Q}_K \sqcup_K CK$ where CK is the cone on K. The path components of X are the singletons $\{a\}$ for $a \in \mathbb{Q}_K\backslash K$ and the set CK. The path component space of X is $\pi_0(X) \cong \mathbb{Q}_K/K$ but the map $q_X \times q_X : X \times X \to \pi_0(X) \times \pi_0(X)$ is not a quotient map [14, §22]. By Prop. 18 the topology of $\pi_0(X \times X)$ is strictly finer than that of $\pi_0(X) \times \pi_0(X)$. □

Other examples of this failure arise in the context of quasitopological fundamental groups [2, 8, 9].

4 Mutliplicative structure

A space X with basepoint x_0 is an *H-space* if there is map $m : X \times X \to X$ such that the map $\mu : \pi_0(X) \times \pi_0(X) \to \pi_0(X)$ given by $\mu([x],[y]) = [m(x,y)]$ is an associative and unital binary operation with unit $[x_0]$ (i.e. $\pi_0(X)$ has the structure of a monoid).

Definition 22. A *semitopological monoid* is a monoid M such that multiplication $M \times M \to M$ is separately continuous (i.e. continuous in each variable). A topological monoid is a monoid M in which multiplication $M \times M \to M$ is (jointly) continuous.

Proposition 23. *If X is an H-space, then $\pi_0(X)$ is a semitopological monoid. If $q_X \times q_X : X \times X \to \pi_0(X) \times \pi_0(X)$ is a quotient map, then $\pi_0(X)$ is a topological monoid.*

Proof. For any $x \in X$, the diagram

$$\begin{tikzcd} X \times \{x\} \arrow[r, "m"] \arrow[d, "q_X"'] & X \arrow[d, "q_X"] \\ \pi_0(X) \times \{[x]\} \arrow[r, "\mu"'] & \pi_0(X) \end{tikzcd}$$

commutes. Since the left vertical arrow is quotient, the bottom map is continuous. Thus right multiplication by $[x]$ is continuous. A similar argument shows that left multiplication by $[x]$ is continuous and that multiplication is jointly continuous when $q_X \times q_X$ is quotient. □

Proposition 24. *For an H-space X, the following are equivalent:*

1. $q_X \times q_X : X \times X \to \pi_0(X) \times \pi_0(X)$ *is quotient.*
2. *The canonical bijection* $\Phi : \pi_0(X \times X) \to \pi_0(X) \times \pi_0(X)$, $\Phi([(x,y)]) = ([x],[y])$ *is a homeomorphism.*
3. $\pi_0(X \times X)$ *is a topological monoid (with its canonical monoid structure).*

Proof. 1. $\Leftrightarrow$ 2. follows from Corollary 19.
1. $\Rightarrow$ 3. If $q_X \times q_X$ is quotient and $\pi_0(X)$ is a topological monoid by Prop. 23. Since 2. also holds, $\pi_0(X \times X)$ is isomorphic to the product topological monoid $\pi_0(X) \times \pi_0(X)$.
3. $\Rightarrow$ 2. Suppose the multiplication $\mu : \pi_0(X \times X) \times \pi_0(X \times X) \to \pi_0(X \times X)$, $\mu([(a, b)], [(c, d)]) = [(ac, bd)]$ where ac and bd are the products under the H-space structure of X. Let $* \in X$ be such that $[*]$ is the identity of $\pi_0(X)$ and $i, j : X \to X \times X$ be the maps given by $i(x) = (*, x)$ and $j(y) = (y, *)$. Let $f = \pi_0(i)$ and $g = \pi_0(j)$. Note that $\mu \circ (f \times g) : \pi_0(X) \times \pi_0(X) \to \pi_0(X \times X)$ is continuous and is precisely the inverse of Φ. □

If X and Y are H-spaces an *H-map* is a map $f : X \to Y$ such that f_0 is a monoid homomorphism. The functorality of π_0 guarantees that such a homomorphism is continuous and thus a morphism of semitopological monoids.

Proposition 25. *π_0 restricts to a functor on the category of H-spaces (and H-maps) to the category of semitopological monoids (and continuous homomorphisms).*

Example 26. *[2]* Since every topological monoid M (with identity e) is an H-space, $\pi_0(M)$ is a semitopological monoid with multiplication $[m][n] = [mn]$. For instance, if Y is any space, let $M(Y) = \coprod_{n \geq 0} Y^n$ denote the free monoid on Y (here $Y^0 = \{e\}$ contains the identity and elements are written as finite words in Y). The path component space $\pi_0(M(Y))$ is a semitopological monoid but is not always a topological monoid itself. In particular, the canonical map $\Phi : \pi_0(M(Y)) \to M(\pi_0(Y))$ is therefore a continuous isomorphism of monoids but is not always a homeomorphism of the underlying spaces.

Definition 27. A space X is *compactly generated* if X has the final topology with respect to all maps $K \to X$ from compact Hausdorff X. In other words C is closed in X iff $f^{-1}(C)$ is closed in K for all compact Hausdorff K and maps $f : K \to X$.

Proposition 28. *If X is a compactly generated H-space such that $X \times X$ and $\pi_0(X) \times \pi_0(X)$ is compactly generated, then $\pi_0(X)$ is a topological monoid.*

Proof. It is well-known that if $q : X \to Y$ is a quotient map where $X \times X$ and $Y \times Y$ are compactly generated, then the product $q \times q : X \times X \to Y \times Y$ is quotient [15]. Thus the assumptions in the stated proposition imply that $q_X \times q_X$ is quotient. The universal property of quotient spaces immediately gives the continuity of multiplication $\pi_0(X) \times \pi_0(X) \to \pi_0(X)$. □

Example 29. All first countable spaces (and more generally sequential spaces) are compactly generated. Therefore if X is a first countable H-space and $\pi_0(X)$ is first countable, then $\pi_0(X)$ is a topological monoid.

Recall an *involution* on a (semitopological) monoid M, with identity e, is a (continuous) function $s : M \to M$ such that $s(e) = e$ and $s(ab) = s(b)s(a)$. Sometimes, we call the pair (M, s) a semitopological monoid with

An *H-involution* on X is a map $*: X \to X$ such that $\pi_0(*): \pi_0(X) \to \pi_0(X)$ is an involution on $\pi_0(X)$. We say X is an *H-group* if there is a map $*: X \to X$ such that $\pi_0(*): \pi_0(X) \to \pi_0(X)$ is an inverse operation which gives $\pi_0(X)$ group structure.

Definition 30. A *quasitopological group* is a group G with topology such that inversion $G \to G$ is continuous and multiplication $G \times G \to G$ is continuous in each variable.

Proposition 31. *If X is an H-group, then $\pi_0(X)$ is a quasitopological group.*

Example 32. If X is a space with basepoint x_0, let $\Omega(X, x_0)$ be the space of loops $\alpha: [0,1] \to X$ based at x_0 with the compact-open topology. The basepoint of $\Omega(X, x_0)$ is typically taken to be the constant loop c at the basepoint. It is well-known that concatenation of loops $\Omega(X, x_0) \times \Omega(X, x_0) \to \Omega(X, x_0)$ gives H-space stucture. Moreover, taking reverse loops $\overline{\alpha}(t) = \alpha(1-t)$ gives an involution $\Omega(X, x_0) \to \Omega(X, x_0)$ and H-group stucture. Therefore the *quasitopological fundamental group*

$$\pi_1(X, x_0) = \pi_0(\Omega(X, x_0))$$

is indeed a quasitopological group. For $n \geq 2$, let $\Omega^n(X, x_0) = \Omega(\Omega^{n-1}(X, x_0), c)$ be the n-th iterated loop space. The *n-th quasitopological homotopy group* of (X, x_0) is the abelian quasitopological group

$$\pi_n(X, x_0) = \pi_0(\Omega^n(X, x_0)).$$

Similarly, one can define relative quasitopological homotopy groups $\pi_n(X, A)$ for subspaces $A \subseteq X$ containing the basepoint which are quasitopological groups for $n \geq 2$.

Remark 33. In general, it is a difficult problem to know when the n-th quasitopological homotopy group is a topological group. There are, however, some simple cases:

1. If $\pi_n(X, x_0)$ is discrete, then clearly it is a topological group.

2. It is easy to check that every finite semitopological monoid (resp. quasitopological group) is a topological monoid (resp. topological group). In particular, if the quasitopological homotopy group $\pi_n(X, x_0)$ is finite, then it must be a topological group.

3. The celebrated Ellis theorem for quasitopological groups [1] gives that every locally compact Hausdorff quasitopological group is a topological group. Thus, if $\pi_n(X, x_0)$ is locally compact Hausdorff, then $\pi_n(X, x_0)$ is a topological group. Typically, however, fundamental groups of Peano continua are large, non-locally compact groups.

4. Often quasitpological homotopy groups fail to satisfy the T_1 separation axiom. If G is a quasitopological group with identity e, then the closure $\overline{e}$ is a closed, normal subgroup of G contained in every open neighborhood

of e. The quotient group $G/\overline{e}$ is the Kolmogorov quotient of G and is thus universal with respect to continuous homomorphisms $f : G \to H$ to T_1 quasitopological groups H. It is easy to check that G is a topological group iff $G/\overline{e}$ is a T_1 quasitopological group. Thus we can conclude that $G = \pi_n(X, x_0)$ is a topological group if $G/\overline{e}$ satisfies one of the previous three conditions.

The situation for countable quasitopological groups is more complicated since for each $n \geq 1$, there are (compact) metric spaces X such that $\pi_n(X, x_0)$ is countable and Hausdorff but fails to be a topological group. On the other hand, there are limits to which quasitopological groups arise as quasitopological homotopy groups. For instance, since the k-th power map $\Omega^n(X, x_0) \to \Omega^n(X, x_0)$, $\alpha \mapsto \alpha^k$ is continuous, so is the power map $\pi_n(X, x_0) \to \pi_n(X, x_0)$, $[\alpha] \mapsto [\alpha]^k$. Since not all quasitopological groups have this property, we conclue that not all quasitopological groups are quasitopological homotopy groups [2].

On the other hand, a countably infinite group with the cofinite topology is a quasitopological group which is not a topological group [1] but which has continuous power maps. Thus to conclude that such groups cannot arise as quasitpological homotopy groups of metric spaces, we must use the fact that multiplication in quasitopological homotopy groups is inherited from continuous multiplication in an H-space.

Lemma 34. *If X is a metric space and $G = \pi_n(X, x_0)$ is countably infinite, then G does not have the cofinite topology.*

Proof. Since X is a metric space $\Omega^n(X, x_0)$ is first countable. Since $G = \pi_0(\Omega^n(X, x_0))$ is first countable by assumption (any countably infinite space with the cofinite topology is first countable), G is a topological group according to Example 29. This contradicts the fact that an infinite group with the cofinite topology is a quasitopological group which is not a topological group. □

It is also interesting to know exactly which quasitopological groups whose underlying group is the additive group of integers can arise as quasitopological homotopy groups.

Proposition 35. *Let G be a non-T_1 quasitopological group whose underlying group is the additive group of integers $\mathbb{Z}$. Then G is a topological group.*

Proof. If the topology of G is indiscrete, clearly G becomes a topological group. Thus we may suppose G is not indiscrete. Since G fails to be T_1, then the closure of the identity is a proper, non-trivial subgroup $n\mathbb{Z} \subset G$. Since $G/n\mathbb{Z}$ is a finite cyclic quasitopological group, it is a topological group according to 2. of Remark 33. Now according to part 4 of Remark 33, G is a topological group. □

Using the previous proposition it is straightforward exercise to classify non-T_1 quasitopological groups with underlying group $\mathbb{Z}$. Lemma 34 indicates the following problem is more challenging.

Problem 36. Characterize (up to isomorphism) the T_1 quasitopological groups with underlying group $\mathbb{Z}$ for which there is a metric space X such that $\pi_n(X, x_0) \cong \mathbb{Z}$ as a quasitopological group.

5 Zero dimensionality and the zeroth shape map

We recall the construction of the zeroth shape space $\check{\pi}_0(X)$ via the Čech expansion. For more details, see [12]. The 1st shape group and map are studied in [4].

The *nerve* of an open cover $\mathscr{U}$ of X is the abstract simplicial complex $N(\mathscr{U})$ whose vertex set is $N(\mathscr{U})_0 = \mathscr{U}$ and vertices $A_0, ..., A_n \in \mathscr{U}$ span an n-simplex iff $\bigcap_{i=0}^{n} A_i \neq \emptyset$. Whenever $\mathscr{V}$ refines $\mathscr{U}$, we can construct a simplicial map $p_{\mathscr{U}\mathscr{V}} : N(\mathscr{V}) \to N(\mathscr{U})$, called a *projection*, given by sending a vertex $V \in N(\mathscr{V})$ to a vertex $U \in \mathscr{U}$ such that $V \subseteq U$. Since we are using refinement, such an assignment of vertices extends linearly to a simplicial map. Moreover, the induced map $|p_{\mathscr{U}\mathscr{V}}| : |N(\mathscr{V})| \to |N(\mathscr{U})|$ on geometric realization is unique up to based homotopy. Thus the continuous function $p_{\mathscr{U}\mathscr{V}0} : \pi_0(|N(\mathscr{V})|) \to \pi_0(|N(\mathscr{U})|)$ induced on discrete (See example 7) path component spaces is independent of the choice of simplicial map.

Let Λ be the subset of $O(X)$ consisting of pairs normal open covers [1] of X. The *zeroth shape homotopy space* is the inverse limit space

$$\check{\pi}_0(X) = \varprojlim \left(\pi_0(|N(\mathscr{U})|), p_{\mathscr{U}\mathscr{V}0}, \Lambda\right).$$

Given an open cover $\mathscr{U}$, a map $p_{\mathscr{U}} : X \to |N(\mathscr{U})|$ is a *canonical map* if $p_{\mathscr{U}}^{-1}(St(U, N(\mathscr{U}))) \subseteq U$ for each $U \in \mathscr{U}$. If $\mathscr{U} \in \Lambda$, such a canonical map is guaranteed to exist: take a locally finite partition of unity $\{\phi_U\}_{U\in\mathscr{U}}$ subordinated to $\mathscr{U}$ and when $U \in \mathscr{U}$ and $x \in U$, determine $p_{\mathscr{U}}(x)$ by requiring its barycentric coordinate belonging to the vertex U of $|N(\mathscr{U})|$ to be $\phi_U(x)$. A canonical map $p_{\mathscr{U}}$ is also unique up to homotopy and whenever $\mathscr{V}$ refines $\mathscr{U}$, the compositions $p_{\mathscr{U}\mathscr{V}} \circ p_{\mathscr{V}}$ and $p_{\mathscr{U}}$ are homotopic as based maps. Therefore the maps $p_{\mathscr{U}0} : \pi_0(X) \to \pi_0(|N(\mathscr{U})|)$ satisfy $p_{\mathscr{U}\mathscr{V}0} \circ p_{\mathscr{V}0} = p_{\mathscr{U}0}$. These homomorphisms induce a canonical map

$$\Psi_0 : \pi_0(X) \to \check{\pi}_0(X) \text{ given by } \Psi_0([x]) = ([p_{\mathscr{U}}(x)])$$

to the limit called the *zeroth shape map*. We say X is π_0-*shape injective* if Ψ_0 is injective.

Definition 37. A space X is *totally separated* if for every distinct $x, y \in X$, there is a clopen set U such that $x \in U$ and $y \in X \backslash U$.

Theorem 38. *Suppose X is paracompact Hausdorff. Then $\Psi_0 : \pi_0(X) \to \check{\pi}_0(X)$ is injective iff $\pi_0(X)$ is totally separated.*

[1] An open cover of X is normal if it admits a partition of unity subordinated to $\mathscr{U}$. Note that every open cover of a paracompact Hausdorff space is normal.

Proof. Recall that $\check{\pi}_0(X)$ is an inverse limit of discrete spaces and is thus totally separated. If $\Psi_0 : \pi_0(X) \to \check{\pi}_0(X)$ is injective, $\pi_0(X)$ continuously injects into a totally separated space and therefore must be totally separated. For the converse, suppose $\pi_0(X)$ is totally separated and $[x_1], [x_2]$ are distinct elements of $\pi_0(X)$. Find a clopen set U_1 such that $[x_1] \in U_1$ and $[x_2] \in U_2 = \pi_0(X)\backslash U_1$. Let $V_i = q_X^{-1}(U_i)$ and $\mathscr{V} = \{V_1, V_2\}$. Note $\mathscr{V}$ is an open cover of X consisting of two disjoint clopen sets and thus $|N(\mathscr{V})|$ consists only of two vertices V_1, V_2. Since $p_{\mathscr{V}}$ is a canonical map, $p_{\mathscr{V}}^{-1}(\{V_i\}) = p_{\mathscr{V}}^{-1}(St(V_i, N(\mathscr{V}))) \subseteq V_i$. But $X = V_1 \cup V_2$ and $X = p_{\mathscr{V}}^{-1}(\{V_2\}) \cup p_{\mathscr{V}}^{-1}(\{V_2\})$ and thus $p_{\mathscr{V}}^{-1}(\{V_i\}) = V_i$. Since $x_i \in V_i$, we have $p_{\mathscr{V}0}([x_1]) = V_1 \neq V_2 = p_{\mathscr{V}0}([x_2])$. By definition of Ψ_0, we have $\Psi_0([x_1]) \neq \Psi_0([x_2])$. □

The following characterization, suggested by the last theorem, has a straightforward proof without using shape theory.

Corollary 39. *A space X is totally separated iff it continuously injects into an inverse limit of discrete spaces.*

Proof. One direction is obvious. If X is totally separated, use Theorem 9 to find a paracompact Hausdorff space $H(X)$ such that $\pi_0(H(X)) \cong X$. By Lemma 38, the natural map $\Psi_0 : X \cong \pi_0(H(X)) \to \check{\pi}_0(H(X))$ is injective. □

Definition 40. A space X is *zero dimensional* if its topology is generated by a basis of clopen sets.

Lemma 41. *Suppose X is paracompact Hausdorff. Then $\Psi_0 : \pi_0(X) \to \check{\pi}_0(X)$ is a topological embedding iff $\pi_0(X)$ is zero dimensional Hausdorff.*

Proof. One direction is obvious. For the converse, suppose $\pi_0(X)$ is zero dimensional Hausdorff. Since all such spaces are totally separated, Ψ_0 is injective. Suppose U is clopen in $\pi_0(X)$. It suffices to show $\Psi_0(U)$ is open in the image of Ψ_0. Suppose $[x] \in U$. Let $V_1 = q_X^{-1}(U)$ and $V_2 = X\backslash V_2$. Then $\mathscr{V} = \{V_1, V_2\}$ is a cover of X by disjoint clopen sets. As in the proof of Theorem 38, we have $p_{\mathscr{V}}(V_i) = V_i$, $i = 1, 2$. Note $[x] \in V_1$ and let

$$W = \{[V_1]\} \times \prod_{\mathscr{U} \neq \mathscr{V}} \pi_0(|N(\mathscr{U})|) \subset \prod_{\mathscr{U}} \pi_0(|N(\mathscr{U})|).$$

We claim that $W \cap Im(\Psi_0) \subseteq \Psi_0(U)$. If $y \in X$ such that $(p_{\mathscr{U}0}([y])) = ([p_{\mathscr{U}}(y)]) \in W$, then $p_{\mathscr{V}}(y) = V_1$ and thus $y \in V_1$. This gives $[y] \in U$ completing the proof. □

The previous theorem suggests the following characterization of zero dimensionality.

Corollary 42. *A space X is zero dimensional Hausdorff iff it embeds as a subspace of an inverse limit of discrete spaces.*

Proof. Recall that inverse limits of discrete spaces are zero dimensional Hausdorff and that this property of spaces is hereditary. Thus one direction is obvious. If X is zero dimensional Hausdorff, then X is a paracompact and thus $\Psi_0 : X \cong \pi_0(X) \to \check{\pi}_0(X)$ is a topological embedding by Lemma 41. □

Corollary 43. *If $n \geq 1$, then $\pi_n(X, x_0)$ is a zero-dimensional quasitopological group iff $\Psi_0 : \pi_n(X, x_0) \to \check{\pi}_0(\Omega^n(X, x_0))$ is an embedding.*

References

[1] A. Arhangel'skii, M. Tkachenko, *Topological Groups and Related Structures*, Series in Pure and Applied Mathematics, Atlantis Studies in Mathematics, 2008.

[2] J. Brazas, The topological fundamental group and free topological groups, Topology Appl. 158 (2011) 779–802.

[3] J. Brazas, *Open subgroups of free topological groups*, arXiv:1209.5486, 2012.

[4] J. Brazas, P. Fabel, *Thick Spanier groups and the first shape group*, To appear in Rocky Mountain J. Math. (2012).

[5] J. Calcut, J. McCarthy, *Discreteness and homogeneity of the topological fundamental group*, Topology Proc. 34 (2009) 339–349.

[6] R. Engelking, *General topology*, Heldermann Verlag Berlin, 1989.

[7] P. Fabel, *The topological hawaiian earring group does not embed in the inverse limit of free groups*, Algebraic & Geometric Topology 5 (2005) 1585–1587.

[8] P. Fabel, *Multiplication is discontinuous in the hawaiian earring group*, Bull. Polish Acad. Sci. Math. 59 (2011) 77–83

[9] P. Fabel, *Compactly generated quasitopological homotopy groups with discontinuous multiplication*, To appear in Topology Proc.

[10] D. Harris, *Every space is a path component space*, Pacific J. Math. 91 (1980) 95–104.

[11] S. MacLane, *Categories for the working mathematician*, GTM, Springer Verlag, 2000.

[12] S. Mardešić and J. Segal, *Shape theory*, North-Holland Publishing Company, 1982.

[13] J.R. Munkres, *Elements of algebraic topology*, Addison-Wesley Publishing Co., Menlo Park, CA., 1984.

[14] J.R. Munkres, *Topology: A first course, 2nd ed.*, Prentice-Hall, Inc., Englewood Cliffs, N.J., 2000.

[15] N.P. Strickland, *The category of CGWH spaces.* Unpublsihed notes.